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Perturbation Methods in a Problem of Waveguide Theory

D. FOX and W. MAGNUS

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ABSTRACT

The reflection coefficient for the basic mode in a widening, straight, two-dimensional waveguide is computed for small wave numbers by using the perturbation method with the electrostatic case as the unperturbed case. Essential features of the treatment are:

(i) the use of conformal mappings,

(ii) the consideration of physically unreal modes, and

(iii) the inversion of an infinite matrix.

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I. The Physical Problem

1. Structure

The problem which we are considering is a two-dimensional one arising from a three-dimensional waveguide structure in which the perfectly conducting surfaces extend from $-\infty$ to ∞ in the direction of the z-axis in a Cartesian coordinate system. The intersection of these conducting surfaces with the x,y-plane is given by the six lines:

$$x \geq 0, \quad y = q\pi \qquad (0 < q < 1)$$

$$x \geq 0, \quad y = -q\pi$$

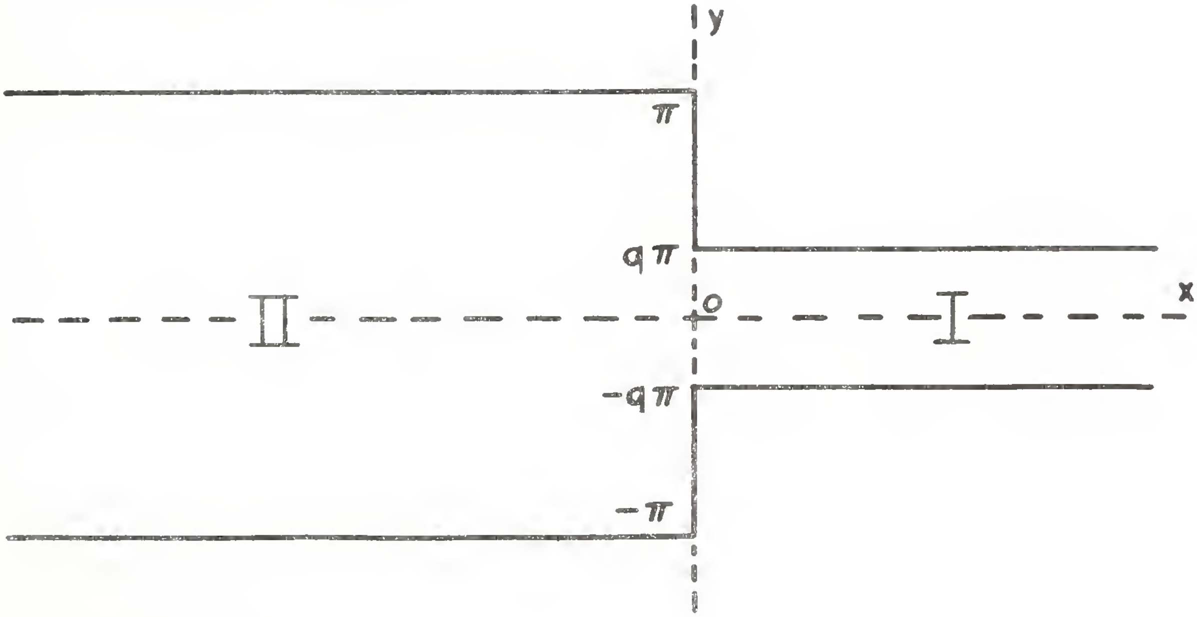
$$x = 0, \quad q\pi \leq y \leq \pi$$

$$x = 0, \quad -q\pi \geq y \geq -\pi$$

$$x \leq 0, \quad y = \pi$$

$$x \leq 0, \quad y = -\pi$$

The segment of the y-axis between $-q\pi$ and $q\pi$, which we shall refer to as the aperture, separates the waveguide structure into two simple regions, I and II.



2. Conditions on the electric field

Let the electric field $\vec{E} = (E_x, E_y, E_z)$, and assume that the time dependency is given by $e^{i\omega t}$ where

$$\omega = kc, \quad k = \text{wave number}, \quad c = \text{velocity of light}.$$

Then we have the conditions

1) E_x, E_y and E_z satisfy $\Delta u + k^2 u = 0$;

2) $\dfrac{\partial E_x}{\partial x} + \dfrac{\partial E_y}{\partial y} + \dfrac{\partial E_z}{\partial z} = 0$;

3) the tangential component of $\vec{E}$ vanishes at the boundaries.

We shall also make the following assumptions:

4) E_y is an even function of y

5) $E_z \equiv 0$.

The behavior of $\vec{E}$ at infinity must satisfy

6) $\lim\limits_{x \to -\infty} |E_y - \tau E_o e^{ikx}| = 0 \;, \quad E_x \to 0$

7) $\lim\limits_{x \to +\infty} |E_y - E_o e^{ikx} - \rho E_o e^{-ikx}| = 0 \;, \quad E_x \to 0$

8) $\iint (E_x^2 + E_y^2)\, dx\, dy$ is bounded over every finite region.

We shall introduce the following definitions:

τ (condition 6)) is called the transmission coefficient.

ρ (condition 7)) is called the reflection coefficient

$E_o e^{ikx}$ is called the incident wave.

$E_y - E_o e^{ikx}$ is called the diffracted wave.

We shall now show that the field is completely determined by E_y alone. Conditions 2) and 5) give

$$2') \quad \frac{\partial E_y}{\partial y} = - \frac{\partial E_x}{\partial x} \quad .$$

This shows that E_y determines E_x up to an arbitrary function of x only. But since E_x and E_y satisfy the reduced wave equation $\Delta u + k^2 u = 0$, the same must be true for both sides of 2'). Furthermore, from 6) and 7) we know that E_x must approach 0 at ∞. This fact gives us the uniqueness we seek. For if 2'), with a fixed E_y, gave rise to two different solutions for E_x, then their difference could only be a function of x alone. If this difference is D(x), then

$$\frac{d^2}{dx^2} D(x) + k^2 D(x) = 0$$

and

$$D(x) \rightarrow 0 \text{ at } \infty.$$

This implies $D(x) \equiv 0$.

Thus, in what follows, we shall limit our attention to the determination of the y-component of the electric field. Our first task is to write the boundary conditions on E_y.

From 1) we have

$$\text{a)}\quad \Delta E_y + k^2 E_y = 0.$$

From 3) we have

$$\text{b)}\quad \frac{\partial E_y}{\partial y}(x, \pm q\pi) = 0 \qquad x > 0$$

$$\text{c)}\quad \frac{\partial E_y}{\partial y}(x, \pm \pi) = 0 \qquad x < 0$$

$$\text{d)}\quad E_y(o,y) = 0 \qquad q\pi < |y| < \pi\ .$$

Note that in b) and c) we used the fact that $E_x = 0$ everywhere on those boundaries. This implied that $\frac{\partial E_x}{\partial x} = 0$, there; and then 2') was used to get b) and c).

Continuity conditions in the aperture are

$$\text{e)}\quad E_y(0^+,y) = E_y(0^-,y) \qquad |y| < q\pi$$

$$\text{f)}\quad \frac{\partial E_y}{\partial x}(0^+,y) = \frac{\partial E_y}{\partial x}(0^-,y) \qquad |y| < q\pi.$$

And from 6) and 7)

$$\text{g)}\quad \lim_{x \to -\infty} |E_y - \tau E_o e^{ikx}| = 0$$

$$\text{h)}\quad \lim_{x \to +\infty} |E_y - E_o e^{ikx} - \rho E_o e^{-ikx}| = 0.$$

II. Method

The first step toward finding the reflection coefficient at the interface is to expand the field in each of the regions (I and II) in Fourier series. The modes are determined on each side by applying all the boundary conditions except the matching condition at the interface. The solution is then assumed to be a series in these modes with constant coefficients.

Applying the matching conditions at the interface gives rise to two infinite sets of linear equations, one arising from matching the fields,and the other from matching the normal derivatives. Using these two sets of equations we can eliminate one group of coefficients, leaving an infinite matrix equation for the other group and a scalar side condition involving the reflection coefficient. Solving this matrix equation, then, will allow us to apply the scalar side condition; and this will yield the value of the reflection coefficient.

The matrix equation which arises from the matching conditions is derived in full and is valid for all values of the wave number, $k < 1$. The coefficients of the expansion of the field in region II are the unknown vector. The equation is solved using a perturbation method. The electrostatic case $(k = 0)$ serves as the unperturbed case. We shall see that the ability to solve the electrostatic case for arbitrary right-hand sides provides us with enough power to determine the higher order terms in the electromagnetic case $(k \neq 0)$. Using this fact, and the scalar side condition, we shall determine the reflection coefficient up to terms of order k^2.

The one remaining facet of the method is the proof of solvability of the electrostatic case for arbitrary right-hand sides. Since the equation in this case is Laplace's equation, we may use conformal mapping as an aid. We map the whole waveguide onto the infinite strip, using the Schwartz-Christoffel formula. This enables us to make use of the fact that each unknown is a Fourier coefficient which can be written as an integral (across the aperture) of the field. The field is written as the derivative of a harmonic potential. In this way the matrix equation may be solved in the electrostatic case for a specific right-hand side. However, this same technique can be used for more general right-hand sides. In order to accomplish this, we introduce into the original problem "unrealistic" modes; i.e., modes which do not die out at $+\infty$. The N^{th} unrealistic mode will give rise to a right-hand side consisting of $N+1$ non-zero entries. Thus, solving for all such unrealistic fields will yield a solution matrix whose product with the original matrix is a triangular matrix. The entries in the triangular matrix can be computed, using the conformal mapping; thus, the original matrix can be inverted. Therefore, we shall have solved the electrostatic case for arbitrary right-hand sides.

III. Infinite System of Linear Equations

In all that follows we shall use the notation that the restriction of E_y to region I is denoted by E_y^{I} and similarly for E_y^{II}. When referring to the entire waveguide we shall write simply E_y. Inasmuch as our ultimate goal is to apply a perturbation technique to this problem, we shall assume that $k < 1$ so that

$$n^2 - k^2q^2 > 0 \qquad \text{for all integers } n \neq 0.$$

By separating variables in a) and applying conditions b) and h) in region I we have

$$E_y^I(x,y) = E_o\left[e^{ikx} + \rho e^{-ikx}\right] + \sum_{n=1}^{\infty} c_n e^{i\frac{k_n}{q}x} \cos\frac{n}{q}y \tag{1}$$

$$\text{where } k_n = i\sqrt{n^2-k^2q^2} \qquad n=1,2,\ldots$$

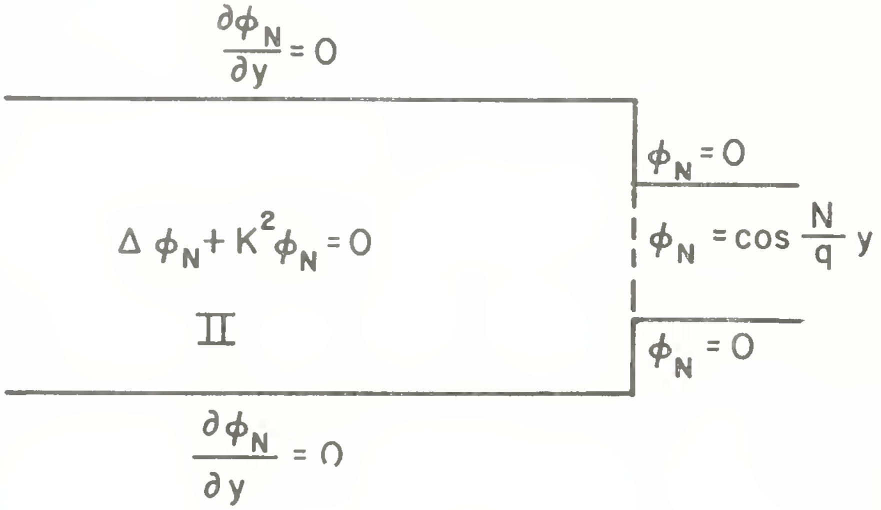
In region II we have the additional condition d) placed on the modes. So we must introduce a set of functions, ϕ_N, satisfying a), c), d) and g). In addition, let the N^{th} function satisfy

$$\text{e}') \quad \phi_N(0^-,y) = \cos\frac{N}{q}y \qquad |y| < q\pi$$

in the aperture.

The reason for condition e') is that the wave coming in from the right is a superposition of cosines in the aperture.

It will be necessary for us to have the ϕ_N explicitly, so we shall derive them here. The N^{th} mode must satisfy the conditions indicated in the sketch below.



$\frac{\partial \phi_N}{\partial y} = 0$
$\phi_N = 0$
$\Delta \phi_N + K^2 \phi_N = 0$
$\phi_N = \cos \frac{N}{q} y$
II
$\phi_N = 0$
$\frac{\partial \phi_N}{\partial y} = 0$

Separating variables, we see that ϕ_N can be expanded in a series of the form

$$\phi_N(x,y) = \sum_{n=0}^{\infty} \alpha_n^{(N)} e^{i \ell_n x} \cos ny \qquad \text{(region II)}$$

$$\text{where } \ell_n = \begin{cases} -i\sqrt{n^2 - k^2} & n \neq 0 \\ k & n = 0 \end{cases}$$

At x = 0 we have

$$\phi_N(o,y) = \sum_{n=0}^{\infty} \alpha_n^{(N)} \cos ny = \begin{cases} \cos \frac{N}{q} y & |y| < q\pi \\ 0 & q\pi < |y| < \pi \end{cases}$$

Consider the case $N \neq 0$, and multiply both sides by cos my and integrate

$$\alpha_o^{(N)} = \frac{1}{\pi} \int_o^{q\pi} \cos \frac{N}{q} y \, dy = 0$$

$$\alpha_m^{(N)} = \frac{2}{\pi} \int_o^{q\pi} \cos my \cos \frac{N}{q} y \, dy .$$

Thus

(2) $$\phi_N(x,y) = \sum_{m=1}^{\infty} S_{mN} e^{i\ell_m x} \cos my \qquad N=1,2,\ldots$$

where

$$S_{mN} = \frac{2}{\pi} \int_o^{q\pi} \cos my \cos \frac{N}{q} y \, dy .$$

Similarly, for N=0, we get

(3) $$\phi_o(x,y) = qe^{ikx} + \sum_{m=1}^{\infty} S_{mo} e^{i\ell_m x} \cos my .$$

The field in region II may now be written

(4) $$E_y^{II}(x,y) = \sum_{n=0}^{\infty} d_n \phi_n(x,y) .$$

Applying condition e), we have

$$E_o(1+\rho) + \sum_{n=1}^{\infty} c_n \cos \frac{n}{q} y = \sum_{m=0}^{\infty} d_m \phi_m(o,y) \qquad |y| < q\pi$$

but, by definition, $\phi_m(o,y) = \cos \frac{m}{q} y \qquad |y| < q\pi$.

Therefore,

$$(5) \qquad E_o(1+\rho) = d_o$$

$$c_n = d_n \qquad n=1,2,\dots \; .$$

Differentiate (1) with respect to x

$$\frac{\partial E_y{}^I(x,y)}{\partial x} = ikE_o \left[e^{ikx} - \rho e^{-ikx} \right] + \frac{i}{q} \sum_{n=1}^{\infty} c_n k_n \, e^{i \frac{k_n}{q} x} \cos \frac{n}{q} y \; .$$

Let $\psi_m(x,y) = \dfrac{\partial \phi_m(x,y)}{\partial x}$.

Condition f) gives

$$ikE_o(1-\rho) + \frac{i}{q} \sum_{n=1}^{\infty} c_n k_n \cos \frac{n}{q} y = \sum_{m=0}^{\infty} d_m \, \psi_m(o,y) \; .$$

Solving for the Fourier coefficients on the left-hand side,

$$(6) \qquad ikE_o(1-\rho) = \frac{1}{q\pi} \int_o^{q\pi} \sum_{m=0}^{\infty} d_m \, \psi_m(o,y) \, dy$$

$$(7) \qquad c_n = \frac{2}{\pi k_n i} \int_o^{q\pi} \sum_{m=0}^{\infty} d_m \, \psi_m(o,y) \cos \frac{n}{q} y \, dy$$

$$n=1,2,\dots \; .$$

Combining (5) with (7),

$$(8) \qquad d_n = \frac{2}{\pi k_n i} \int_o^{q\pi} \sum_{m=0}^{\infty} d_m \, \psi_m(o,y) \cos \frac{n}{q} y \, dy \qquad n=1,2,\ldots \, .$$

From (2) and (3) it can be seen that

$$(9) \qquad \psi_o(x,y) = ikqe^{ikx} + i \sum_{r=1}^{\infty} S_{ro} \, \ell_r \, e^{i\ell_r x} \cos ry$$

$$\psi_m(x,y) = i \sum_{r=1}^{\infty} \ell_r \, S_{rm} \, e^{i\ell_r x} \cos ry \qquad m=1,2,\ldots \, .$$

Therefore,

$$(10) \qquad \sum_{m=0}^{\infty} d_m \, \psi_m(o,y) = ikqd_o + i \sum_{m=0}^{\infty} d_m \sum_{r=1}^{\infty} S_{rm} \, \ell_r \cos ry \, .$$

Substituting into (8),

$$d_n = \frac{2}{\pi k_n} \int_o^{q\pi} \left\{ kqd_o + \sum_{m=0}^{\infty} d_m \sum_{r=1}^{\infty} \ell_r \, S_{rm} \cos ry \right\} \cos \frac{n}{q} y \, dy$$

$$n=1,2,\ldots$$

$$(11) \quad d_n = \sum_{m=0}^{\infty} d_m \sum_{r=1}^{\infty} \frac{\ell_r}{k_n} S_{rm} \, S_{rn} \qquad n=1,2,\ldots \, .$$

Equation (11) is an infinite system of linear equations in the unknown Fourier coefficients, d_n.

A scalar side condition is obtained by substituting (10) into (6).

$$ikE_o(1-\rho) = \frac{1}{q\pi}\int_0^{q\pi}\left\{ikqd_o + i\sum_{m=0}^{\infty} d_m \sum_{r=1}^{\infty} S_{rm}\,\ell_r \cos ry\right\}dy$$

$$(12) \qquad = ikqd_o + \frac{i}{q\pi}\sum_{m=0}^{\infty} d_m \sum_{r=1}^{\infty} \frac{1}{r} S_{rm}\,\ell_r \sin r\pi q \ .$$

It can be seen from equations (3) and (5) that the transmission coefficient, τ, is given by

$$(13) \qquad \tau = q(1+\rho),$$

which gives a simple relation between the reflection and transmission coefficient. Using the value of d_o found in (5), we have from (12)

$$(14) \qquad \sum_{m=0}^{\infty} d_m \sum_{r=1}^{\infty} \frac{\ell_r}{q\pi r} S_{rm} \sin q\pi r = kE_o(1-\rho-q-\rho q).$$

This scalar condition will be used to evaluate ρ after (11) has been solved for the d_n's.

IV. Perturbation Method

In order to use the perturbation method in the solution of the equations of the previous section, we must establish their explicit dependence upon k. Having done that, we shall solve the electrostatic problem (k=0) explicitly and use it as the unperturbed case.

Equation (11) depends upon k only in its term, $\frac{\ell_r}{k_n}$.

$$\frac{\ell_r}{k_n} = -\frac{\sqrt{r^2-k^2}}{\sqrt{n^2-k^2q^2}} = -\frac{r}{n}\frac{\sqrt{1-\frac{k^2}{r^2}}}{\sqrt{1-\frac{k^2q^2}{n^2}}} = -\frac{r}{n}\sqrt{\left(1-\frac{k^2}{r^2}\right)\left(1+\frac{k^2q^2}{n^2}+\ldots\right)}$$

$$= -\frac{r}{n}\sqrt{1-\left(\frac{k^2}{r^2}-\frac{k^2q^2}{n^2}\right)+\ldots}$$

$$= -\frac{r}{n}+\frac{r}{2n}\left(\frac{k^2}{r^2}-\frac{k^2q^2}{n^2}\right)+\ldots$$

$$= -\frac{r}{n}+\frac{k^2}{2}\left(\frac{1}{nr}-\frac{q^2r}{n^3}\right)+\ldots .$$

Equation (14) depends upon k only in its term, ℓ_r.

$$\ell_r = -i\sqrt{r^2-k^2} = -ir\sqrt{1-\frac{k^2}{r^2}} = -ir+\frac{ik^2}{2r}+\ldots$$

Equation (11) can be written nicely in matrix notation as follows:

$$\vec{d} = (d_0, d_1, d_2, \ldots)$$

$$T = \text{a matrix of elements } T_{nr} = \frac{r}{n} S_{rn} \qquad n \neq 0$$

$$= 0 \qquad n = 0$$

$$S = \text{matrix of elements } S_{rn} = \frac{2}{\pi} \int_0^{q\pi} \cos ry \cos \frac{n}{q} y \, dy$$

(as previously defined)

$$U = \text{matrix of elements } U_{nr} = -\frac{1}{2}\left(\frac{1}{r^2} - \frac{q^2}{n^2}\right) T_{nr} \qquad n, r \neq 0$$

$$= 0 \qquad n \text{ or } r = 0 \,.$$

Now we add equation (5) to equation (11) and use the notation that $\vec{1}$ is the vector

$$\vec{1} = \{1, 0, 0, 0, \ldots\} \,.$$

We assume that all summations go from 0 to $+\infty$.

$$\vec{d} = -\{(T + k^2 U)S\} (\vec{d}) + E_0(1+\rho)\, \vec{1} \tag{15}$$

If I is the identity matrix, we have

$$\{I + TS + k^2 US\} (\vec{d}) = E_0(1+\rho)\, \vec{1} \,. \tag{16}$$

Let $T^*_{rm} = -\frac{i}{q\pi} S_{rm} \sin q\pi r \qquad r=0,1,2,\ldots$

$$U^*_{rm} = -\frac{1}{2r^2} T^*_{rm} \qquad r=1,2,\ldots$$

$$= 0 \qquad r=0 \,.$$

Then equation (14) becomes

$$(17) \qquad \sum_{m=0}^{\infty} d_m \sum_{r=0}^{\infty} (T^*_{rm} + k^2 U^*_{rm}) = kE_o(1-\rho-q-\rho q)$$

For shorthand, let us define vectors $\vec{t}^*$ and $\vec{u}^*$ whose components are:

$$t^*_m = \sum_{r=0}^{\infty} T^*_{rm}$$

$$u^*_m = \sum_{r=0}^{\infty} U^*_{rm} \quad .$$

Then (17) can be rewritten as

$$(18) \qquad (\vec{t}^* + k^2 \vec{u}^*)\cdot\vec{d} = kE_o(1-\rho-q-\rho q) \; .$$

Equations (16) and (18) form the system which we shall solve. First we shall solve (16) using the perturbation method to invert the matrix. Then we shall use the solution vector, $\vec{d}$, in (18) to compute the reflection coefficient, ρ.

The method will proceed as follows. We shall show that the electrostatic system

$$\left\{I + TS\right\} (\vec{d}) = \vec{R}$$

can be solved for arbitrary right-hand sides, $\vec{R}$, in such a way that $\vec{t}^* \cdot \vec{d} = 0$. When this is known, the full system can be solved up to terms of order k^2 by setting

$$\vec{d} = \vec{e} + k^2\vec{f} \ .$$

Equation (16) becomes

$$\left\{I + TS + k^2US\right\} \ (\vec{e} + k^2\vec{f}) = E_o(1+\rho)\vec{1} \ .$$

Thus

$$\left\{I + TS\right\} (\vec{e}) + k^2 \left\{I + TS\right\} \ (\vec{f}) + k^2 \left\{US\right\} \ (\vec{e}) = E_o(1+\rho)\vec{1} \ .$$

We now can find a unique vector, $\vec{e}$, such that

$$\left\{I + TS\right\} (\vec{e}) = \vec{1}$$

(which implies that $E_o(1+\rho)\vec{e}$ solves the electrostatic system) and $\vec{t}^* \cdot \vec{e} = 0$. Now, having found $\vec{e}$, we compute $\left\{US\right\} (\vec{e})$ and solve

$$\left\{I + TS\right\} (\vec{f}) = - \left\{US\right\} (\vec{e})$$

in such a way that $\vec{t}^* \cdot \vec{f} = 0$.

Having found $\vec{d}(=\vec{e} + k^2\vec{f})$ we substitute this solution vector into (18)

$$(\vec{t}^* + k^2\vec{u}^*) \cdot (\vec{e} + k^2\vec{f}) = \frac{kE_o(1-\rho-q-\rho q)}{E_o(1+\rho)}$$

But $\vec{t}^* \cdot \vec{e} = \vec{t}^* \cdot \vec{f} = 0$.

Therefore,

$$k^2 \vec{u}^* \cdot \vec{e} = \frac{kE_o(1-\rho-q-\rho q)}{E_o(1+\rho)}$$

$$(19) \qquad \rho = \frac{1-(q+k\vec{u}^* \cdot \vec{e})}{1+(q+k\vec{u}^* \cdot \vec{e})} \quad .$$

This equation gives the value of the reflection coefficient for small values of wave number.

V. Electrostatic Case

We have the system of linear equations

$$\left\{I + TS\right\} (\vec{d}\,) = \vec{R}$$

and we wish to show that it can be solved for arbitrary right-hand sides in such a way that the solution vector, $\vec{d}$, satisfies the scalar condition

$$\vec{t}^* \cdot \vec{d} = 0 \ .$$

To this end we first invert the matrix $\left\{I + TS\right\}$. The method we shall use is to produce a matrix, D, and a triangular matrix, Δ, such that

$$\left\{I + TS\right\} \left\{D\right\} = \Delta \ .$$

Since Δ is triangular, its inverse can easily be computed; thus, the full solution is

$$\left\{I + TS\right\}^{-1} = D\,\Delta^{-1} \ .$$

First we shall construct the triangular matrix, Δ. To this end, let us consider the field in region I. It has the form

$$E_y^{I}(x,y) = \sum_{n=0}^{\infty} c_n e^{-\frac{n}{q}x} \cos \frac{n}{q} y \tag{20}$$

in the electrostatic case. If we relax the condition at $+\infty$ and allow the field to become exponentially large, new modes may be introduced. We shall refer to these as "unrealistic" modes. Let the N^{th} unrealistic mode be denoted by $E_N^{I}(x,y)$

$$E_N^{I}(x,y) = \sum_{n=-N}^{\infty} c_n^N e^{-\frac{n}{q}x} \cos \frac{n}{q} y . \tag{21}$$

In region II we have

$$E_N^{II}(x,y) = \sum_{n=0}^{\infty} d_n^N \phi_n(x,y) \tag{22}$$

where, in the electrostatic case,

$$\begin{aligned} \phi_o(x,y) &= q + \sum_{m=1}^{\infty} S_{mo} e^{mx} \cos my \\ \phi_n(x,y) &= \sum_{m=1}^{\infty} S_{mn} e^{mx} \cos my && n \neq 0 \\ \psi_n(x,y) = \frac{\partial \phi_n}{\partial x} &= \sum_{m=1}^{\infty} m\, S_{mn} e^{mx} \cos my && \text{all } n . \end{aligned} \tag{23}$$

Equations (21) and (22) provide us with an infinitude of new fields. Solution of these will yield the matrices, D and Δ, which we seek.

Matching (21) and (22) in the aperture x=0, $|y| < q\pi$

$$\sum_{n=-N}^{\infty} c_n^N \cos \frac{n}{q} y = \sum_{n=0}^{\infty} d_n^N \phi_n(o,y) .$$

This implies that

$$c_n^N = d_n^N \qquad n > N$$

$$(24) \quad c_n^N + c_{-n}^N = d_n^N \qquad 0 < n \leq N$$

$$c_o^N = d_o^N$$

since $\cos \frac{n}{q} y = \cos \left(-\frac{n}{q}\right) y$

and $\phi_n(o,y) = \cos \frac{n}{q} y , \quad |y| < q\pi .$

Let us define

$$c_n^+ = \begin{cases} c_n^N & n > 0 \\ \frac{1}{2} c_o^N & n = 0 \end{cases}$$

$$c_n^- = \begin{cases} 0 & n > N \\ c_{-n}^N & 0 < n \leq N \\ \frac{1}{2} c_o^N & n = 0 \end{cases}$$

Then

$$d_n^N = c_n^+ + c_n^- \text{ , for all n.} \tag{25}$$

We apply the derivative condition

$$\frac{\partial E_N^I(x,y)}{\partial x} = -\frac{1}{q}\sum_{m=-N}^{\infty} m\, c_m^N\, e^{-\frac{m}{q}x} \cos\frac{m}{q}y$$

$$\frac{\partial E_N^{II}(x,y)}{\partial x} = \sum_{n=0}^{\infty} d_n^N\, \psi_n(x,y)$$

$$\sum_{m=1}^{\infty} \frac{m}{q}(c_m^- - c_m^+) \cos\frac{m}{q}y = \sum_{n=0}^{\infty} d_n^N\, \psi_n(o,y) \qquad |y| < q\pi \text{ .} \tag{26}$$

Multiply by $\cos\frac{n}{q}y$ and integrate

$$\frac{n}{q}(c_n^- - c_n^+) = \frac{2}{q\pi}\int_o^{q\pi} \sum_{m=0}^{\infty} d_m^N\, \psi_m(o,y) \cos\frac{n}{q}y\, dy \text{ .}$$

Multiply this by $\frac{q}{n}$ and add equation (25)

$$2c_n^- = d_n^N + \sum_{m=0}^{\infty} d_m^N \frac{2}{\pi n}\int_o^{q\pi} \psi_m(o,y) \cos\frac{n}{q}y\, dy \text{ .} \tag{27}$$

However, it is easily seen that (27) is merely

$$(28) \qquad \left\{I + TS\right\} (\vec{d}_N) = 2\vec{c}_N^{\,-}$$

where $\vec{d}_N = (d_o^N, d_1^N, \ldots)$

and $\vec{c}_N^{\,-} = (\frac{1}{2}c_o^N, c_{-1}^N, \ldots, c_{-N}^N, 0, 0\ldots)$.

If for each N we can compute $\vec{c}_N^{\,-}$ and $\vec{d}_N$, we will have found the matrices D and Δ. The solution vectors, $\vec{d}_N$, will comprise D; and the right-hand sides, $\vec{c}_N^{\,-}$, will comprise the triangular matrix, Δ.

Thus, we must do the following for each N

(i) Compute the vector $\vec{c}_N^{\,-}$

(ii) Find the solution vector $\vec{d}_N$.

Problem (i) may be solved as follows:

We map the waveguide structure onto the strip $|v| \leq \pi$ in the $w=u+iv$ plane. The Schwartz-Christoffel formula gives us the mapping implicitly

$$(29) \qquad z = \log \left[\frac{\left(\sqrt{q^2e^w+1} + q\sqrt{e^w+1}\right)^{2q} e^w \left(1-q^2\right)^{1-q}}{\left(\sqrt{q^2e^w+1} + \sqrt{e^w+1}\right)^2} \right]$$

$$\frac{dw}{dz} = \sqrt{\frac{e^w+1}{q^2e^w+1}}$$

Let $\zeta = e^{-\frac{z}{q}}$, $\quad \sigma = e^{-w}$.

Substituting into (29)

$$(30) \qquad \zeta = \sigma \frac{\left(\sqrt{q^2+\sigma} + \sqrt{1+\sigma}\right)^{\frac{2}{q}}}{\left(\sqrt{q^2+\sigma} + q\sqrt{1+\sigma}\right)^2 \left(1-q^2\right)^{\frac{1-q}{q}}}$$

Equation (30) shows that σ can be expanded in a power series in ζ. The field E_N can be written as the real part of the derivative of the potential, e^{Nw}.

Let $\Phi_N = \frac{d}{dz} e^{Nw}$

so that $E_N = \text{Re}\ \Phi_N$.

We assert that Φ_N can be expanded in a series of the type

$$c_{-N}^{(N)} e^{\frac{Nz}{q}} + c_{1-N}^{(N)} e^{\frac{(1-N)}{q}z} + \ldots + c_o^{(N)} + \sum_{\ell=1}^{\infty} c_\ell^{(N)} e^{-\frac{\ell}{q}z}$$

$$= \sum_{\ell=-N}^{\infty} c_\ell^{(N)} \zeta^\ell \quad .$$

This form follows from (30) and from

$$\Phi_N = Ne^{Nw} \frac{dw}{dz} = N \sigma^{-N} \sqrt{\frac{1+\sigma}{q^2+\sigma}} \quad .$$

The residue theorem asserts that

$$c_{-\ell}^{(N)} = \frac{1}{2\pi i} \oint_{\zeta=0} \Phi_N \zeta^{\ell-1} d\zeta \qquad 1 \leqq \ell \leqq N$$

This reduces to

$$(31) \qquad c_{-\ell}^{(N)} = \frac{1}{2\pi i} \frac{N}{q\left(1-q^2\right)^{(1-q)\frac{\ell}{q}}} \oint_{\sigma=0} \sigma^{\ell-N-1} \frac{\left(\sqrt{q^2+\sigma} + \sqrt{1+\sigma}\right)^{2\frac{\ell}{q}}}{\left(\sqrt{q^2+\sigma} + q\sqrt{1+\sigma}\right)^{2\ell}} d\sigma$$

This verifies that $c_{-\ell}^{(N)} = 0, \quad \ell > N.$

Problem (ii) may be solved by using the fact that in the aperture each component of the vector $\vec{d}_N$ is a coefficient of a Fourier series in the aperture.

$$(32) \qquad d_m^N = \frac{1}{q\pi} \int_{-q\pi}^{q\pi} E_N \cos \frac{m}{q} y \, dy \; .$$

But $E_N = \text{Re}\, \Phi_N = \text{Re}\, Ne^{Nw} \frac{dw}{dz}$.

Writing the full complex integral (and dropping the "Re" for now),

$$d_m^N = \frac{1}{2\pi qi}\int_{-q\pi i}^{q\pi i} N\, e^{Nw}\,\frac{dw}{dz}\left\{e^{\frac{m}{q}z} + e^{-\frac{m}{q}z}\right\} dz$$

$$= \frac{N}{2\pi qi}\int_{-\pi i+\log(\frac{1}{q^2})}^{\pi i+\log(\frac{1}{q^2})} e^{Nw}\left\{e^{\frac{m}{q}F(w)} + e^{-\frac{m}{q}F(w)}\right\} dw$$

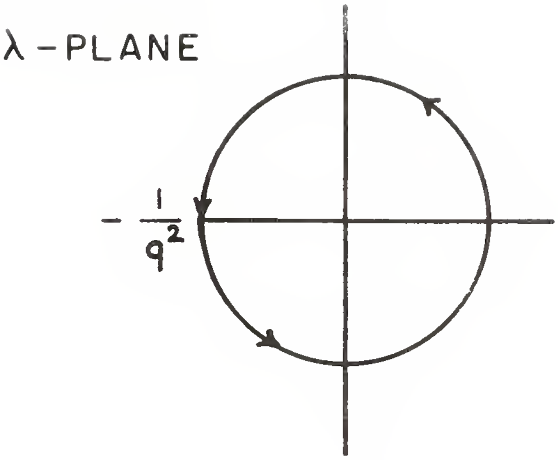
where $F(w)$ is the Schwartz-Christoffel mapping given in equation (29).

Using the substitution $\lambda = e^w$

$$(33)\quad d_m^N = \mathrm{Re}\,\frac{N}{2\pi qi}\left[(1-q^2)^{(1-q)\frac{m}{q}}\int \lambda^{N-1+\frac{m}{q}}\frac{\left(\sqrt{q^2\lambda+1} + q\sqrt{\lambda+1}\right)^{2m}}{\left(\sqrt{q^2\lambda+1} + \sqrt{\lambda+1}\right)^{2\frac{m}{q}}}\,d\lambda\right.$$

$$\left. + (1-q^2)^{(q-1)\frac{m}{q}}\int \lambda^{N-1-\frac{m}{q}}\frac{\left(\sqrt{q^2\lambda+1} + \sqrt{\lambda+1}\right)^{\frac{2m}{q}}}{\left(\sqrt{q^2\lambda+1} + q\sqrt{\lambda+1}\right)^{2m}}\,d\lambda\right]$$

The path of integration is around a circle of radius $\frac{1}{q^2}$ in the positive sense, as illustrated:



It remains now to show that solutions of the system

$$\left\{ I + TS \right\} (\vec{d}) = \vec{R} \qquad \text{satisfy}$$

$$\vec{t}^{\,*} \cdot \vec{d} = 0 \, .$$

The reason for this is that the components d_m of $\vec{d}$ are the Fourier coefficients of the derivative of a potential function Re $\Phi = \frac{\partial v}{\partial y}$ in the aperture.

$$\vec{t}^{\,*} \cdot \vec{d} = \frac{1}{2q\pi} \int_{-q\pi}^{q\pi} \frac{\partial^2 v(o,y)}{\partial y \partial x} \, dy = \frac{\partial v}{\partial x} \Bigg|_{(o,-q\pi)}^{(o,q\pi)} \tag{34}$$

But v is constant on both boundaries, so that

$$\frac{\partial v}{\partial x} = 0 \quad \text{there.}$$

Therefore, $\vec{t}^{\,*} \cdot \vec{d} = 0$.

VI. Convergence

It has been tacitly assumed in the previous computations that the convergence of the various infinite series is good enough to justify the operations performed. For example, the final formula for the reflection coefficient (equation (19)) involves an infinite series represented by the dot product, $\vec{u}^* \cdot \vec{e}$. We shall show here that $\vec{u}^* \cdot \vec{e}$ does converge and, in fact, that it converges absolutely.

From equation (17) it can be seen that

$$(35) \qquad \vec{u}^* \cdot \vec{e} = \sum_{m=0}^{\infty} e_m \sum_{r=1}^{\infty} \frac{i}{2q\pi r^2} \sin q\pi r \, \frac{2}{\pi} \int_0^{q\pi} \cos ry \cos \frac{m}{q} y \, dy$$

$$= \frac{i}{q\pi^2} \sum_{m=0}^{\infty} e_m \left[\sum_{r=1}^{\infty} \frac{\sin q\pi r}{r^2} \int_0^{q\pi} \cos ry \cos \frac{m}{q} y \, dy \right]$$

We shall first show that the factor in brackets is less than a constant times $\frac{1}{m}$ for m large.

It can be seen that

$$\int_0^{q\pi} \cos ry \cos \frac{m}{q} y \, dy = (-1)^m \sin r\pi q \, \frac{r}{r^2 - \frac{m^2}{q^2}} \, .$$

In the special case of $q = \frac{m}{r}$ the above formula still has validity, as can be seen by a passage to the limit. Thus, we shall show that

$$2 \sum_{r=1}^{\infty} \frac{1}{r} \frac{1}{r^2 - \frac{m^2}{q^2}} \sin^2 q\pi r$$

behaves like $\frac{1}{m}$ for large m.

We break this sum up as follows:

$$S_1 = \sum_{r=1}^{\infty} \frac{1}{r^2} \frac{1}{r + \frac{m}{q}} \sin^2 r\pi q$$

$$S_2 = \sum_{r=1}^{\left[\frac{m}{2q}\right]} \frac{1}{r^2} \frac{1}{r - \frac{m}{q}} \sin^2 r\pi q$$

$$S_3 = \sum_{r=\left[\frac{m}{2q}\right]+1}^{\left[\frac{3m}{2q}\right]} \frac{1}{r^2} \frac{1}{r - \frac{m}{q}} \sin^2 r\pi q$$

$$S_4 = \sum_{r=\left[\frac{3m}{2q}\right]+1}^{\infty} \frac{1}{r^2} \frac{1}{r - \frac{m}{q}} \sin^2 r\pi q$$

where $[x]$ = integral part of x.

$$S_1 \leq \frac{1}{\frac{m}{q}} \sum_{r=1}^{\infty} \frac{1}{r^2} = k_1 \frac{1}{m}$$

$$|S_2| \leq \frac{1}{\frac{m}{2q}} \sum_{r=1}^{\left[\frac{m}{2q}\right]} \frac{1}{r^2} < \frac{2q}{m} \sum_{r=1}^{\infty} \frac{1}{r^2} = k_2 \frac{1}{m}$$

$$S_4 \leq \frac{1}{\frac{m}{2q}} \sum_{r=\left[\frac{3m}{2q}\right]+1}^{\infty} \frac{1}{r^2} = k_4 \frac{1}{m}$$

Finally, $|S_3| = \left| \sum_{r=\left[\frac{m}{2q}\right]+1}^{\left[\frac{3m}{2q}\right]} \frac{1}{r^2} \frac{1}{r - \frac{m}{q}} \sin^2 r\pi q \right|$

Let $\epsilon_r = r - \frac{m}{q}$, so that

$$-\frac{m}{2q} \leq \epsilon_r \leq \frac{m}{2q} .$$

For each r,

$$\left| \frac{1}{r - \frac{m}{q}} \sin^2 r\pi q \right| = \frac{1}{|\epsilon_r|} \sin^2(m\pi + q\pi\epsilon_r)$$

$$= \frac{\sin^2 q\pi\epsilon_r}{|\epsilon_r|} .$$

This expression is clearly bounded, independent of m. Call this bound B. We have

$$|S_3| \leq B \sum_{r=\left[\frac{m}{2q}\right]+1}^{\left[\frac{3m}{2q}\right]} \frac{1}{r^2} < B \frac{1}{\left(\frac{m}{2q}\right)^2} \cdot m = k_3 \frac{1}{m} .$$

Thus, we have shown that the original sum is less than $(k_1 + k_2 + k_3 + k_4) \frac{1}{m}$ as m becomes large.

In order to complete the proof of convergence of $\vec{u}^* \cdot \vec{e}$ (equation (35)), we must demonstrate the absolute convergence of

$$(36) \qquad \sum_{m=1}^{\infty} \frac{1}{m} e_m .$$

Each component of the vector $\vec{e}$ is a coefficient of the Fourier expansion of the field in the aperture.

$$(37) \qquad e_m = \frac{1}{q\pi} \int_{-q\pi}^{q\pi} E_y \cos \frac{m}{q} y \, dy .$$

Inasmuch as E_y vanishes for $q\pi < |y| < \pi$, equation (37) can be written as

$$(38) \qquad e_m = \frac{1}{q\pi} \int_{-\pi}^{\pi} E_y \cos \frac{m}{q} y \, dy$$

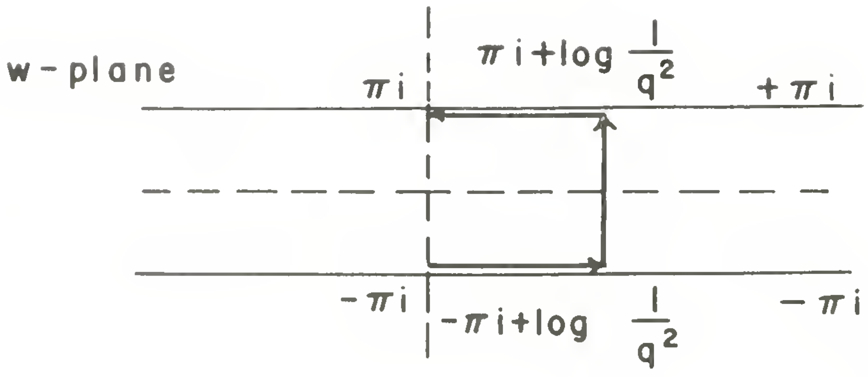
$$= \frac{1}{2\pi q i} \int_{-\pi i}^{\pi i} \frac{dw}{dz} \left(e^{\frac{m}{q} z} + e^{-\frac{m}{q} z} \right) dz \ .$$

We can convert this to a line integral in the w-plane using the Schwartz-Christoffel mapping given in equation (29).

$$(39)\quad e_m = \frac{1}{2\pi q i} \left\{ (1 - q^2)^{(1-q)\frac{m}{q}} \int_C e^{\frac{m}{q} w} \frac{\left(\sqrt{q^2 e^w + 1} + q\sqrt{e^w + 1}\right)^{2m}}{\left(\sqrt{q^2 e^w + 1} + \sqrt{e^w + 1}\right)^{\frac{2m}{q}}} dw \right.$$

$$\left. + \frac{1}{\left(1-q^2\right)^{(1-q)\frac{m}{q}}} \int_C e^{-\frac{m}{q} w} \frac{\left(\sqrt{q^2 e^w + 1} + \sqrt{e^w + 1}\right)^{\frac{2m}{q}}}{\left(\sqrt{q^2 e^w + 1} + q\sqrt{e^w + 1}\right)^{2m}} dw \right\}$$

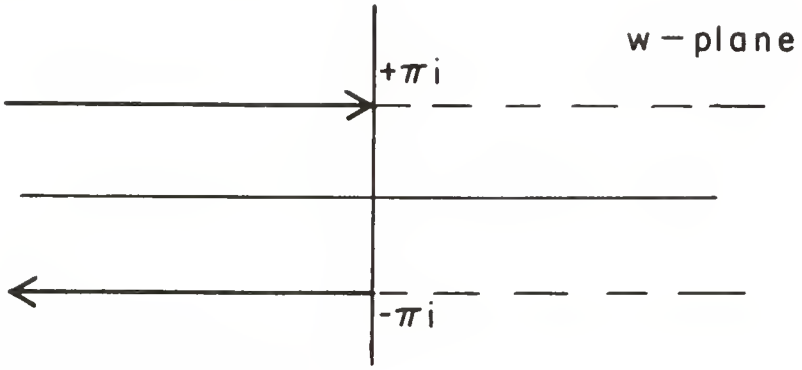
The path of integration, C, is illustrated in the following sketch:



The first integral,

$$(40) \qquad \frac{1}{2\pi q i}\,(1-q^2)^{(1-q)\frac{m}{q}} \int_C e^{\frac{m}{q}w}\, \frac{\left(\sqrt{q^2e^w+1} + q\sqrt{e^w+1}\right)^{2m}}{\left(\sqrt{q^2e^w+1} + \sqrt{e^w+1}\right)^{\frac{2m}{q}}}\, dw$$

will converge if the path is altered in such a way that $\mathrm{Re}(w) < 0$; so we shall change the path into



Along the bottom path we have

$$w = -\pi i - s \qquad\qquad \text{as } s \to \infty .$$

Therefore, the absolute value of the integral is

$$\frac{1}{2\pi q}\,(1-q^2)^{(1-q)\frac{m}{q}} \int_0^\infty e^{-\frac{m}{q}s}\, \frac{\left(\sqrt{1-q^2e^{-s}} + q\sqrt{1-e^{-s}}\right)^{2m}}{\left(\sqrt{1-q^2e^{-s}} + \sqrt{1-e^{-s}}\right)^{\frac{2m}{q}}}\, ds$$

But, for $s > 0$,

$$\sqrt{1-q^2} \le \left(\sqrt{1-q^2e^{-s}} + q\sqrt{1-e^{-s}}\right) \le \left(\sqrt{1-q^2e^{-s}} + \sqrt{1-e^{-s}}\right)$$

This means that the absolute value is less than or equal to

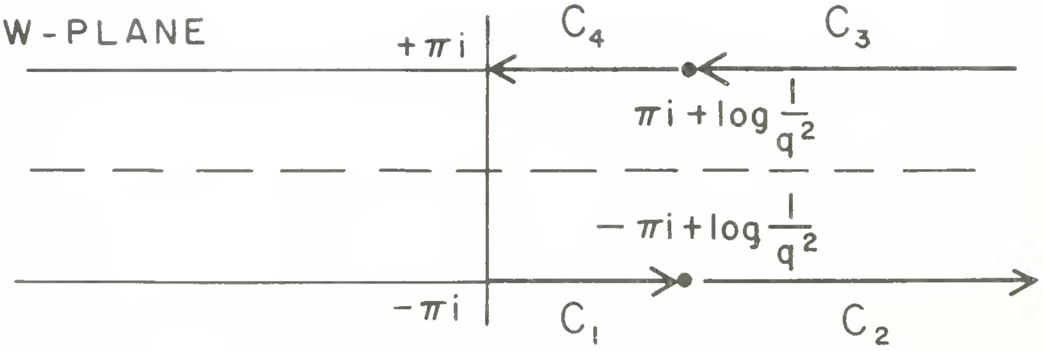
$$\frac{1}{2\pi q}\,(1-q^2)^{(1-q)\frac{m}{q}} \int_0^\infty e^{-\frac{m}{q}s} \left(\sqrt{1-q^2e^{-s}} + \sqrt{1-e^{-s}}\right)^{2m-\frac{2m}{q}} ds$$

$$\le \frac{1}{2\pi q}\,(1-q^2)^{(1-q)\frac{m}{q}} \int_0^\infty \left(\sqrt{1-q^2}\right)^{2m-\frac{2m}{q}} e^{-\frac{m}{q}s}\, ds$$

$$= \frac{1}{2\pi m}\,.$$

The same analysis goes through on the upper path of integration, and this shows the absolute convergence of the first half of (36).

For the second integral (39) we can break up the path as follows:



Along C_2, $w = -\pi i + s$ and we have

$$(41)\quad \frac{1}{2\pi q i\left(1-q^2\right)^{(1-q)\frac{m}{q}}}\int_{\log\frac{1}{q^2}}^{\infty} e^{-m(-\pi i+s)}\,\frac{\left(\sqrt{q^2-e^{-s}}+\sqrt{1-e^{-s}}\right)^{\frac{2m}{q}}}{\left(\sqrt{q^2-e^{-s}}+q\sqrt{1-e^{-s}}\right)^{2m}}\,ds$$

$$= \frac{(-1)^m}{2\pi q i\left(1-q^2\right)^{(1-q)\frac{m}{q}}}\int_{\log\frac{1}{q^2}}^{\infty} e^{-ms}\,\frac{\left(\sqrt{q^2-e^{-s}}+\sqrt{1-e^{-s}}\right)^{\frac{2m}{q}}}{\left(\sqrt{q^2-e^{-s}}+q\sqrt{1-e^{-s}}\right)^{2m}}\,ds$$

Along C_3, $w = \pi i + s$,

$$(42)\quad \frac{1}{2\pi q i\left(1-q^2\right)^{(1-q)\frac{m}{q}}}\int_{\infty}^{\log\frac{1}{q^2}} e^{-m(\pi i+s)}\,\frac{\left(\sqrt{q^2-e^{-s}}+\sqrt{1-e^{-s}}\right)^{\frac{2m}{q}}}{\left(\sqrt{q^2-e^{-s}}+q\sqrt{1-e^{-s}}\right)^{2m}}\,ds$$

$$= -\frac{(-1)^m}{2\pi q i\left(1-q^2\right)^{(1-q)\frac{m}{q}}}\int_{\log\frac{1}{q^2}}^{\infty} e^{-ms}\,\frac{\left(\sqrt{q^2-e^{-s}}+\sqrt{1-e^{-s}}\right)^{\frac{2m}{q}}}{\left(\sqrt{q^2-e^{-s}}+q\sqrt{1-e^{-s}}\right)^{2m}}\,ds$$

But, a comparison of (41) and (42) (along C_2 and C_3) shows that they cancel each other out exactly.

We are left, then, with integrals along C_1 and C_4. On C_4 we have

$$w = \pi i + s \qquad 0 \leq s \leq \log \frac{1}{q^2}$$

$$(43) \qquad \frac{1}{2\pi q i \left(1-q^2\right)^{(1-q)\frac{m}{q}}} \int_{\log \frac{1}{q^2}}^{0} e^{-m(\pi i+s)} \frac{\left(\sqrt{q^2-e^{-s}}+\sqrt{1-e^{-s}}\right)^{\frac{2m}{q}}}{\left(\sqrt{q^2-e^{-s}}+q\sqrt{1-e^{-s}}\right)^{2m}} \, ds$$

$$= \frac{(-1)^{m+1}}{2\pi q i} \int_{0}^{\log \frac{1}{q^2}} \frac{\left(\sqrt{\frac{1-e^{-s}}{1-q^2}} + i\sqrt{\frac{e^{-s}-q^2}{1-q^2}}\right)^{\frac{2m}{q}}}{\left(q\sqrt{\frac{e^{s}-1}{1-q^2}} + i\sqrt{\frac{1-q^2e^{s}}{1-q^2}}\right)^{2m}} \, ds$$

Note that in the interval both numerator and denominator are of absolute value 1. We introduce

$$(44) \qquad \begin{aligned} &\cos\alpha = \sqrt{\frac{1-e^{-s}}{1-q^2}}, \quad \sin\alpha = \sqrt{\frac{e^{-s}-q^2}{1-q^2}}, \quad \frac{d\alpha}{ds} = -\frac{1}{2}\frac{e^{-s}}{\sqrt{e^{-s}-q^2}\,\sqrt{1-e^{-s}}}, \\ &\cos\beta = q\sqrt{\frac{e^{s}-1}{1-q^2}}, \quad \sin\beta = \sqrt{\frac{1-q^2e^{s}}{1-q^2}}, \quad \frac{d\beta}{ds} = -\frac{q}{2}\frac{e^{s}}{\sqrt{e^{s}-1}\sqrt{1-q^2e^{s}}} \end{aligned}$$

$$0 \leq \alpha,\beta \leq \frac{\pi}{2}$$

$$(45) \qquad \gamma = \frac{\alpha}{q} - \beta \, , \qquad\qquad \frac{d\gamma}{ds} = -\frac{1}{2}\tan\beta$$

$$0 \leq \gamma \leq \frac{\pi}{2}\left(\frac{1}{q} - 1\right)$$

Thus, we have that the integral along C_4 is

$$(46) \qquad \frac{(-1)^{m+1}}{2\pi qi} \int_0^{\log\frac{1}{q^2}} e^{2mi\gamma}\, ds \, .$$

In the same way the integral along C_1 can be shown to be

$$(47) \qquad \frac{(-1)^{m}}{2\pi qi} \int_0^{\log\frac{1}{q^2}} e^{-2mi\gamma}\, ds \, .$$

Combining (46) and (47) yields the single integral

$$(48) \qquad \frac{(-1)^{m+1}}{\pi q} \int_0^{\log\frac{1}{q^2}} \sin\,(2m\gamma)\, ds \, .$$

We now wish to change the variable of integration from s to γ. Equation (45) shows that γ is monotonic; but it also shows that near $\gamma=0$ ($s = \log\frac{1}{q^2}$), the integral will become improper. Writing

$$\omega_0 = \frac{\pi}{2}\left(\frac{1}{q} - 1\right)$$

and letting ω_1 be a fixed number between 0 and ω_o, we break up (48) as follows

$$(49)\quad \frac{(-1)^{m+1}}{\pi q}\left\{\int_{\omega_o}^{\omega_1}\left[-\sin(2m\gamma)2\cot\beta\right]d\gamma + \int_{\omega_1}^{o}\sin(2m\gamma)\frac{ds}{d\gamma}\,d\gamma\right\}$$

For any fixed value of $\omega_1 > 0$, $\cot\beta$ is an analytic function in the interval ω_o to ω_1. Thus, we may integrate by parts to show that the first integral is of order $\frac{1}{m}$.

For the second integral we must go back to equation (43). We are near the point $s = \log\frac{1}{q^2}$, so we set

$$t = s - \log\frac{1}{q^2}$$

$$\text{and } 1 + R = \frac{\left(\sqrt{\frac{1-e^{-s}}{1-q^2}} + i\sqrt{\frac{e^{-s}-q^2}{1-q^2}}\right)^{\frac{1}{q}}}{\left(q\sqrt{\frac{e^s-1}{1-q^2}} + i\sqrt{\frac{1-q^2e^s}{1-q^2}}\right)}$$

$1 + R$ can then be expanded in a power series in t, near t=0. This computation shows that

$$(50)\quad 1 + R = 1 - a_1 t^{3/2} + \text{higher order terms in } t^{1/2}.$$

But $e^{2mi\gamma} = (1 + R)^{2m}$. Therefore,

$$(51) \qquad \gamma = -i \log (1 + R) = -iR + \text{higher terms} .$$

Combining (51) with (50),

$$(52) \qquad \gamma = b_1 t^{3/2} + \text{higher terms.}$$

Then $\gamma^{1/3} = b_1^{1/3} t^{1/2} +$ higher terms. This mapping can be inverted in some neighborhood of the origin to get

$$t^{1/2} = c_1 \gamma^{1/3} + \text{higher terms in } \gamma^{1/3} .$$

This implies that

$$(53) \qquad t = c_1^2 \gamma^{2/3} + \text{higher terms.}$$

Thus

$$(54) \qquad ds = dt = \left[d_1 \gamma^{-\frac{1}{3}} + f(\gamma^{\frac{1}{3}}) \right] d\gamma$$

where f is an analytic function near $\gamma = 0$.

We are now in a position to return to the second integral of (49). From the foregoing discussion it can be seen that ω_1 should be chosen small enough to be within the radii of convergence of the various power series we have just used. That being the case, we have

$$(55)\qquad \int_{\omega_1}^{0} \frac{\sin(2m\gamma)}{\gamma^{1/3}}\, d\gamma + \int_{\omega_1}^{0} \sin(2m\gamma)\, f(\gamma^{1/3})\, d\gamma .$$

The second integral is easily seen to be of order $\frac{1}{m}$, and the first can be evaluated as follows. It is well known that

$$(56)\qquad \int_{0}^{\infty} \frac{\sin yx}{x^{\nu}}\, dx = \Gamma(1-\nu) \cos\frac{\pi\nu}{2} \frac{1}{y^{1-\nu}} \qquad y > 0, \quad 0 < \nu < 2 .$$

Now

$$\int_{\omega_1}^{0} = \int_{\omega_1}^{\infty} - \int_{0}^{\infty} .$$

By (56),

$$\int_{0}^{\infty} \frac{\sin(2m\gamma)}{\gamma^{1/3}}\, d\gamma = \text{const}\, \frac{1}{m^{2/3}} .$$

Further,

$$(57)\qquad \int_{\omega_1}^{\infty} \frac{\sin(2m\gamma)}{\gamma^{1/3}}\, d\gamma = - \frac{\cos(2m\gamma)}{2m\gamma^{1/3}} \Bigg|_{\omega_1}^{\infty} - \frac{1}{6m} \int_{\omega_1}^{\infty} \frac{\cos(2m\gamma)}{\gamma^{4/3}}\, d\gamma .$$

The latter integral now converges absolutely. This shows that expression (57) is of order $\frac{1}{m}$.

Combining all these results shows that $\vec{u}^{*}\cdot\vec{e}$ converges absolutely. For we have shown that this sum is at worst like

$$\sum_{m} \frac{1}{m^{5/3}}$$

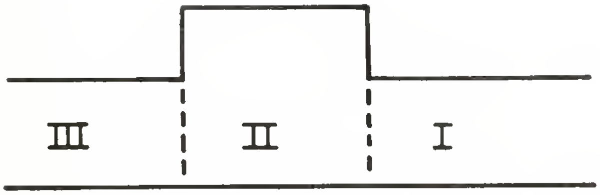
which is absolutely convergent.

VII. Concluding Remarks

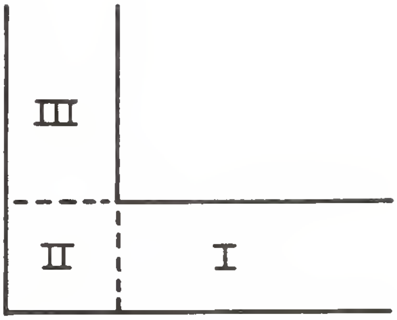
We have restricted our investigations to a first order approximation of the value of the reflection coefficient. Higher order approximations are available, since we have the inverse of the matrix determining the electrostatic case.

The method of solution used in this paper may be tedious, but it is sufficiently general to be applicable to a wide range of waveguide problems. For example, it can be used in finding the field in more complex waveguide structures such as those illustrated in the following sketches:

A



B



In each of these problems there are three distinct regions, and this will give rise to two sets of reflection and transmission coefficients. (At first it might be thought that regions II and III in problem B could be combined into a single region, but elementary analysis shows that a field of this form is over-determined and can yield only the trivial solution.)

A problem treated by Magnus and Oberhettinger[1] is essentially equivalent to our problem in the case where $q = \frac{1}{2}$. However, the method used there is based on algebraic identities rather than on conformal mappings.

REFERENCES

[1] Magnus, W. and F. Oberhettinger	-	On Systems of Linear Equations in the Theory of Guided Waves - Comm. on Pure and Appl. Math.; 3, 393 (1950).
[2] Whittaker, E. T. and G. N. Watson	-	A Course of Modern Analysis - Cambridge, London, 1958.

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